

Nonlinear Schrödinger equation from generalized exact uncertainty principle

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Abstract. Inspired by the generalized uncertainty principle (GUP), which adds gravitational effects to the standard description of quantum uncertainty, we extend the exact uncertainty principle (EUP) approach by Hall and Reginatto [J. Phys. A: Math. Gen. (2002) **35** 3289], and obtain a (quasi)nonlinear Schrödinger equation. This quantum evolution equation of unusual form, enjoys several desired properties like separation of non-interacting subsystems or plane-wave solutions for free particles. Starting with the harmonic oscillator example, we show that every solution of this equation respects the gravitationally-induced minimal position uncertainty proportional to the Planck length. Quite surprisingly, our result successfully merges the core of classical physics with non-relativistic quantum mechanics in its extremal form. We predict that the commonly accepted phenomenon, namely a modification of a free-particle dispersion relation due to quantum gravity might not occur in reality.

1. Introduction

Even though, the famous Schrödinger equation does not provide the most general description of quantum systems (for instance, only approximates the Dirac equation) it remains useful while studying fundamental aspects of quantum mechanics. A prominent example aspect discussed in this contribution is the *linearity of Quantum Mechanics* (superposition principle) which, if valid universally, shall also apply to the Schrödinger equation in its pure form. We thus do not consider here the Gross-Pitaevski (describing Bose-Einstein condensate) or Schrödinger-Newton (self-gravity effects in Newtonian approximation) extensions of the standard non-relativistic quantum dynamics, but focus on a nonlinearity *per se*, possibly present in the genuine Schrödinger equation.

In the literature (see a rare example [1] not related to Bose-Einstein condensate) on the field one can mainly find the discussion of the quadratic (in the wavefunction $\psi(x, t)$; by x we denote an n -dimensional position vector) nonlinearity included as an additional term proportional to the probability density $\rho = |\psi|^2$. Most of formal research focuses on the mathematical aspects of this basic nonlinearity (integrability, blow up, etc. [2–4]), while more interdisciplinary approaches apply the resulting wave equation beyond quantum physics [5, 6]. In a more general scenario, the nonlinearity is introduced by a function of ρ , commonly (but not always) being equal to 0 for $\rho = 0$ (see [3] for few interesting examples). Note that in order to mimic the structure of the linear equation, every discussed correction is always multiplied by the wavefunction, so that all nonlinear contributions play the role of state (or only density) dependent potentials.

As pointed out by Bialynicki-Birula [7], nonlinear modifications of the above type (except one):

- (i) Introduce an extra interaction between separable subsystems,
- (ii) Spoil the standard normalization procedure for stationary states.

While the first issue can sometimes be accepted as an emanation of a possibly unavoidable link between subsystems one intends to separate, the second issue diminishes the beauty of mathematical analogy between quantum states and projective rays of the Hilbert space. In the comprehensive discussion devoted to formally reasonable nonlinearities [8], the second argument also referred to as lack of homogeneity, led to the conclusion that all inhomogeneous proposals are actually not of physical relevance. On the other hand some homogeneous nonlinearities, like $(|\nabla\psi|/|\psi|)^2$ discussed by Kibble [9], bring on board the third issue, namely they

- (iii) Violate Galilean invariance.

The single form of nonlinearity, depending only on the density and free from the above limitations, is given by the logarithmic term $-b \log \rho$. It was long ago shown [10–12] that the parameter b if different from zero must be very small, at least $b < 4 \times 10^{-10} \text{eV}$. Experimental tests [13] put a more rigorous limitation $b < 3.3 \times 10^{-15} \text{eV}$ (for a short summary of other obtained estimations see [7]). It is worth noticing that the logarithmic correction was later derived on the ground of the stochastic equation [14].

It is rather obvious that possible nonlinearities of the Schrödinger equation (if any) must in normal conditions contribute in a negligible manner. On the other hand, in extraordinary situations (eg. very high energies), when the conceivable nonlinearities could play any noticeable role, the Schrödinger equation will likely be an insufficient approximation. Nevertheless, the question whether the pure Schrödinger equation contains any nonlinearities (even of extremely low contribution) remains of fundamental interest, as it challenges the superposition principle. Moreover, nonlinear Schrödinger dynamics can be interesting from the perspective of down-to-earth problems such as quantum state discrimination [15].

In the current paper, instead of *ad hoc* proposing a new form of a suitable nonlinearity, we follow and slightly generalize the approach of *exact uncertainty principle* (EUP) by Hall and Reginatto [16–20]. In their seminal paper [16], the authors have postulated a fundamental scaling relation between nonclassical momentum fluctuations and uncertainty in position (please refer to Section 2 for details), which is a prerequisite for the Heisenberg uncertainty relation (HUR), $\Delta x \Delta p \geq \hbar/2$, involving position and momentum standard deviations.

Our proposed generalization is driven by a reverted reasoning. In the regime relevant for potential nonlinearities of quantum dynamics, also the HUR might likely require a modification. A model example of this theoretically predicted phenomenon is the family of generalized uncertainty principles (GUP) with its most basic member given by [21–24]

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[1 + \beta (\Delta p)^2 \right]. \quad (1)$$

The right hand side of the above inequality contains corrections depending on the momentum uncertainty, introduced by the parameter $\beta = \beta_0 l_p^2 / \hbar^2 \equiv \beta_0 G / (\hbar c^3)$ encoding gravitational effects. $l_p = 1.62 \times 10^{-35} \text{m}$ denotes the Planck length, G is the Newton constant while β_0 is a numerical parameter (likely of order of unity) depending on the approach towards quantization of gravity. In this contribution I however would not like to discuss more and rely on the theoretical foundations of the GUP (interested readers shall consult the comprehensive review [24]), since violation of the Lorentz invariance present in the modified energy-momentum dispersion relations or in doubly special relativity does not belong to the set of commonly accepted laws of nature. I find it sufficient to expect, that due to various possible physical reasons such as potential existence of maximal proper acceleration [25] (note that this preserves Lorentz invariance [26]), the equation (1) or a similar one replaces the HUR [27]. Let me only observe that the mathematical structure of (1) and its counterparts, leads by a straightforward optimization to the minimal measurable length [23], or in other cases to existence of minimal measurable momentum, minimal time interval or maximal measurable energy [28]. Such constrains, when considered, influence various physical predictions such as description of black holes [29] or Newton's laws of gravity [30].

As we will see later, since (1) is sharper than the HUR, a suitable modification of the EUP is a natural path to follow. The exact uncertainty principle, adjusted to a sharper uncertainty relation (like the GUP, but not only), necessarily brings (quasi)nonlinear corrections to the Schrödinger equation, which in a quite general scenario becomes ($V(x)$ is a potential):

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V - \frac{\hbar^2}{2m} \sum_{l=1}^n W(C F_l[\rho]) \frac{1}{|\psi|} \frac{\partial^2 |\psi|}{\partial x_l^2} \right) \psi. \quad (2)$$

$W(\cdot)$ is a dimensionless function associated with the uncertainty relation considered (please see Eqs. 30 and 34 in Section 3) and expected to assume values significantly smaller than 1. By construction, it has a property $W(0) = 0$. The functional

$$F_l[\rho] = \int d^n x \frac{1}{\rho} \left(\frac{\partial \rho}{\partial x_l} \right)^2, \quad (3)$$

is the Fisher information with respect to the variable x_l , while $C = \hbar^2/4$.

I call the above equation quasi-nonlinear because of the specific forms of nonlinearity appearing in its last term. First of all, the dependence on ρ via the function W is hidden inside the space integrals present in $F_l[\rho]$. This means, that while looking for stationary states

$$\psi(x, t) = e^{-iEt/\hbar} \psi(x), \quad (4)$$

for which ρ is time-independent, one can treat $W_l \equiv W(CF_l[\rho])$ as being a collection of fixed parameters, and attempt to find the solution in the form $\psi(x; W_1, \dots, W_n)$. Then, one is left with a system of n algebraic equations for the W_l parameters, given by the consistency conditions utilizing the definition (3) and the function W , namely:

$$W_l = W\left(CF_l[|\psi(x; W_1, \dots, W_n)|^2]\right). \quad (5)$$

Moreover, even though $|\psi| = \sqrt{\rho}$, in the common case when $\psi(x)$ from (4) is real, the derivative term reduces to the standard (linear) form with the second derivative of $\psi(x)$. In this scenario we simply recover the usual, time-independent Schrödinger equation with each component of the Laplacian multiplied by the constant $1 + W_l$ (effective mass). We thus obtain an almost linear situation with the single exception, that at the end the consistency equations (5) shall be solved.

Remarkably, the nonlinear Eq. 2 enjoys all three desired properties, namely separability of non-interacting systems, the norm invariance (homogeneity) and Galilean invariance. If one substitutes:

$$\psi(x, t) = \prod_{l=1}^n \psi_l(x_l, t), \quad (6)$$

then $F_l[\rho] = F_l[\rho_l]$, with $\rho_l = |\psi_l|^2$, and $|\psi|^{-1} \partial_t^2 |\psi|$ depends only on $|\psi_l|$. The full, time-dependent dynamics of the state (6) becomes separable, provided the potential contains no interaction between subsystems. Any Galilean transformation shifts the time derivative and adds a position-dependent phase to the wavefunction. The Fisher information is invariant as the functional of the density, while the second-derivative terms depend on $|\psi|$ (again the phase does not contribute). In relation to the norm invariance for the stationary states (4), if $\psi(x; W_1, \dots, W_n)$ is a solution valid before the application of the consistency conditions (5), then also $A\psi(x; W_1, \dots, W_n)$ is the proper solution in the same case. One can normalize this solution in a standard way, so that the constant A acquires a dependence on W_l . Actually, the algebraic conditions (5) can be written down and possibly solved only after the normalization procedure.

Obviously the true dynamics spoils the norm invariance letting the nonlinearity have a genuine character. Since the superposition principle does not hold, the stationary states do not provide a frame to study the time evolution. Note also, what is to be expected, that Eq. 2 reduces to the pure Schrödinger equation if the solution is a plane wave (in this case $F_l[\rho] = 0$).

In the next section we shall bring on board the basic results concerning the approach based on the exact uncertainty principle merged with classical dynamics. In Section 3 we generalize the EUP in order to handle the modified forms of the HUR and derive the corresponding quantum dynamics. In the last section we discuss the Gaussian solution of Eq. 2 for the 1D harmonic potential, with special emphasis on the GUP case.

2. Quantum dynamics from exact uncertainty principle

We begin this short review of the EUP formalism by a list of all relevant quantities and concepts. Except an expanded discussion (see Sec. 2.1) of technical assumptions relevant for the momentum fluctuations, followed by their mild modifications, the material presented below is a summary of sections 2 and 3 from [16]. The starting point is the classical dynamics described in terms of the Hamilton's principal function $S(x, t)$ and the probability density $P(x, t)$. The function S evolves according to the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + V = 0, \quad (7)$$

and its coordinate derivatives (components of ∇S) give the classical momentum \mathbf{p}_{cl} . The probability density satisfies the continuity equation

$$\frac{\partial P}{\partial t} + \nabla \cdot \left(P \frac{1}{m} \nabla S \right) = 0, \quad (8)$$

since $m^{-1} \nabla S$ is a velocity field. Both equations can be derived by virtue of a variational principle applied to the classical action

$$\mathcal{A}_C = \int dt d^n x P \left[\frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + V \right]. \quad (9)$$

The quantum dynamics (standard Schrödinger equation) emerges from the classical action modified by a term brought by the EUP. To make a long story short (for all other details see [16]) the quantum momentum is described as a fluctuating classical momentum: $\mathbf{p} = \mathbf{p}_{\text{cl}} + \mathbf{N}$, or componentwise ($l = 1, \dots, n$)

$$p_l = \frac{\partial S}{\partial x_l} + N_l. \quad (10)$$

The fluctuation term \mathbf{N} satisfies certain randomness assumptions (see Sec. 2.1) and, as a result, modifies the action to the quantumly corrected form

$$\mathcal{A}_Q = \mathcal{A}_C + \frac{1}{2m} \sum_{l=1}^n \int dt (\Delta N_l)^2, \quad (11)$$

with each ΔN_l being the root-mean-square fluctuation in l th direction, averaged over the n -dimensional coordinate space.

2.1. Properties of momentum fluctuations

The extra ingredient of the above formalism, namely the momentum fluctuations \mathbf{N} , shall be subject to physically motivated restrictions. To this end, the authors of [16] distinguished two types of averages. Given a random, position-dependent quantity $\Phi(x)$ one can either only average out the fluctuations at a given point — obtaining the position-dependent field $\bar{\Phi}(x)$ — or go one step further and average the resulting field over the position space:

$$\langle \Phi \rangle = \int d^n x P(x) \bar{\Phi}(x). \quad (12)$$

Reasonably behaving fluctuations shall on average vanish in every point [16], that is $\bar{\mathbf{N}} = 0$. The authors of [16] observed that for the discussed purpose it is sufficient to impose a couple of substantially weaker conditions (see Eq. 5 from [16])

$$\langle \mathbf{N} \rangle = 0, \quad \langle \nabla S \cdot \mathbf{N} \rangle \equiv \langle \mathbf{p}_{\text{cl}} \cdot \mathbf{N} \rangle = 0. \quad (13)$$

These two conditions also have an appealing meaning, namely the first one says that fluctuations disappear on the total average $\langle \cdot \rangle$ while the second one imposes *unbiasedness* between the fluctuations in question and the classical momentum.

Eq. 11 appears as a consequence of the replacement

$$\int d^n x P(x) \nabla S \cdot \nabla S \equiv \langle \mathbf{p}_{\text{cl}} \cdot \mathbf{p}_{\text{cl}} \rangle \mapsto \langle \mathbf{p} \cdot \mathbf{p} \rangle = \langle (\mathbf{p}_{\text{cl}} + \mathbf{N}) \cdot (\mathbf{p}_{\text{cl}} + \mathbf{N}) \rangle, \quad (14)$$

and

$$\langle (\mathbf{p}_{\text{cl}} + \mathbf{N}) \cdot (\mathbf{p}_{\text{cl}} + \mathbf{N}) \rangle = \langle \mathbf{p}_{\text{cl}} \cdot \mathbf{p}_{\text{cl}} \rangle + 2 \langle \mathbf{p}_{\text{cl}} \cdot \mathbf{N} \rangle + \langle \mathbf{N} \cdot \mathbf{N} \rangle, \quad (15)$$

where $\langle \mathbf{N} \cdot \mathbf{N} \rangle = (\Delta N)^2$ are the *total fluctuations*:

$$\Delta N = \sqrt{\sum_{l=1}^n (\Delta N_l)^2}. \quad (16)$$

It is important to understand that mathematically (to derive Eq. 11), one needs to eliminate the mixed term $\langle \mathbf{p}_{\text{cl}} \cdot \mathbf{N} \rangle$. This goal, however, can be achieved in many ways. One can impose the stronger condition $\bar{\mathbf{N}} = 0$ which works because \mathbf{p}_{cl} is free from fluctuations, or by resorting to the unbiasedness assumption which literally says that the mixed term in question vanishes. Both ways are physically reasonable, and can thus be used interchangeably. For instance, in [18] only the stronger condition is utilized.

Staying on the purely mathematical ground, we could find more possibilities leading to the same result. For example, one can consider the momentum being a complex field, such that the classical momentum is real while the momentum fluctuations are purely imaginary. In the replacement (14) one would then need to use a complex conjugate field as well, $\langle \mathbf{p}_{\text{cl}} \cdot \mathbf{p}_{\text{cl}} \rangle \mapsto \langle \mathbf{p}^* \cdot \mathbf{p} \rangle$. In this way, the “desired” result follows for any (imaginary) \mathbf{N} and without randomness assumptions. One shall argue that introducing the complex numbers would spoil the classical flavor of the whole derivation. On the other hand Quantum Mechanics relies on the complex wave vectors and functions, while the momentum and other observables are Hermitian (though mildly complex) operators.

In the current contribution we shall not further explore the path of complex replacement, but we will rely on the randomness assumptions presented above. We would like to observe, however, that the single unbiasedness assumption is not enough from the perspective of the HUR. One could easily find that

$$(\Delta p)^2 = \langle \mathbf{p} \cdot \mathbf{p} \rangle - \langle \mathbf{p} \rangle \cdot \langle \mathbf{p} \rangle = \langle \mathbf{p}_{\text{cl}} \cdot \mathbf{p}_{\text{cl}} \rangle + 2 \langle \mathbf{p}_{\text{cl}} \cdot \mathbf{N} \rangle + \langle \mathbf{N} \cdot \mathbf{N} \rangle - \langle \mathbf{p}_{\text{cl}} \rangle \cdot \langle \mathbf{p}_{\text{cl}} \rangle, \quad (17)$$

where $\langle \mathbf{p} \rangle = \langle \mathbf{p}_{\text{cl}} \rangle$ as $\langle \mathbf{N} \rangle = 0$. Due to the unbiasedness assumption, we obtain $\Delta p \geq \Delta N$, which as discussed in the next subsection is the prerequisite to the HUR.

The single unbiasedness assumption is too weak to render the componentwise relations $\Delta p_l \geq \Delta N_l$ satisfied independently for all $l \in \{1, \dots, n\}$. Since these inequalities are necessary to assure the validity of the individual HURs (for any position-momentum couple) we shall strengthen the unbiasedness condition from (13). Based on the above discussion we assume that

$$\bar{\mathbf{N}} = 0 \quad \text{or} \quad \langle \mathbf{N} \rangle = 0 \quad \text{and} \quad \forall_l \langle \partial_l S \cdot \mathbf{N}_l \rangle = 0. \quad (18)$$

This unified condition allows that the momentum fluctuations either vanish on average in the strong sense, or only in the weak sense being componentwise unbiased with the classical momentum.

2.2. The exact uncertainty principle

Still following [16], we denote by δx a “direct measure of uncertainty in position”. The EUP states that δx is fully characterized by P and, most importantly, that the total fluctuations are inversely correlated with δx . In other words, if one considers a rescaling transformation $P(x) \mapsto P_\kappa(x) = \kappa^n P(\kappa x)$, then since $P_\kappa(x)$ is narrower (broader) for $\kappa > 1$ ($\kappa < 1$), the position uncertainty accordingly transformed as $\delta x \mapsto \kappa^{-1} \delta x$ becomes smaller (bigger) in comparison with the initial one. The inverse law of the EUP implies the linear transformation of the total momentum fluctuations

$$\Delta N \mapsto \kappa \Delta N. \quad (19)$$

Further analysis of the above scaling relation [16] leads to the solution for $(\Delta N)^2$ of the form:

$$(\Delta N)^2 = C \sum_{l=1}^n F_l [P], \quad C = \hbar^2/4. \quad (20)$$

Note that in the case of independent subsystems, the particular fluctuations associated with every direction shall also be mutually independent. This property together with (16) and (20) uniquely fix $\Delta N_l = \sqrt{CF_l [P]}$. It is also important to emphasize here the spherical symmetry related to the rescaling transformation. In fact, not only the total fluctuations become multiplied by κ , but the same scaling property independently applies to every ΔN_l . As a result, $(\Delta N)^2$ is invariant with respect to rotations of the coordinate space, and as a functional can only be made with invariant quantities such as $\nabla P \cdot \nabla P$. This property has been implicitly used in [16] as the starting point in the derivation of (20).

Finally, if one *decides* to describe the position uncertainty in the l th direction by the square root of the inverse of the Fisher information, $\delta x_l = 1/\sqrt{F_l [P]}$, one finds the equality of the form

$$\delta x_l \Delta N_l = \frac{\hbar}{2}. \quad (21)$$

Note that δx_l is a good, though not required, choice for the direct uncertainty measure δx in the 1D case. From the well known Cramer-Rao bound one has $\Delta x_l \geq \delta x_l$ while, as explained in Sec. 2.1, $\Delta p_l \geq \Delta N_l$. As a corollary from the EUP one thus obtains the HUR, satisfied independently for all $l \in \{1, \dots, n\}$.

The Euler-Lagrange equations for the quantum action \mathcal{A}_Q reproduce the continuity equation (8) and lead to the modified Hamilton-Jacobi equation of the form

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + \frac{\hbar^2}{8m} \left[\frac{1}{P^2} \nabla P \cdot \nabla P - \frac{2}{P} \nabla^2 P \right] + V = 0. \quad (22)$$

Finally, by a subtle substitution

$$S = -i\hbar \ln \frac{\psi}{\sqrt{P}}, \quad \text{equivalently given as} \quad \psi = \sqrt{P} e^{iS/\hbar}, \quad (23)$$

one transforms (22) to be the pure Schrödinger equation:

$$-i\hbar \frac{1}{\psi} \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \frac{1}{\psi} \nabla^2 \psi + V = 0. \quad (24)$$

From (23) one also gets that $P = |\psi|^2 \equiv \rho$.

3. Generalized exact uncertainty principle

Having in mind the GUP, Eq. (1), as a model generalization of the HUR, we would like to study its possible interrelations with the EUP. First of all, if a small value of the scaling parameter κ is considered, the momentum fluctuations remain small and the term $(\Delta p)^2$ present on the right hand side of (1) does not play a significant role. For bigger values of κ , however, as the momentum fluctuations are assumed to grow linearly with κ , the more visible right hand side contribution might violate the GUP. Since the term proportional to $(\Delta p)^2$ is multiplied by a very small parameter, in the regime in which it actually makes a physical sense to discuss (1), the quadratic contribution shall remain majorized by the basic linear term from the left hand side. To make this mechanism work it will be desired to let the momentum fluctuations grow superlinearly. The overall message

is thus that if generalized (sharper) forms of the quantum uncertainty relation are expected instead of the HUR, the EUP shall deviate from the law of inverse correlation.

To follow the intuition sketched above, let us modify the EUP scaling law (19) to the form

$$\Delta N_l \mapsto w^{-1}(\kappa w(\Delta N_l)), \quad (25)$$

or

$$w(\Delta N_l) \mapsto \kappa w(\Delta N_l), \quad (26)$$

with κ presented as the proper scaling factor. The difference with respect to (19) is due to a non-negative, increasing function $w(\cdot)$ such that:

$$\lim_{z \rightarrow 0} w(z) = 0, \quad \lim_{z \rightarrow 0} w'(z) = 1, \quad (27)$$

and (most importantly)

$$w(z) < z, \quad \text{whenever} \quad z > 0. \quad (28)$$

The first requirement from (27) assures that the new scaling transformation does not generate fluctuations if they are initially absent. The second condition states that for infinitesimal values of ΔN_l the formula (25) does not differ from the original EUP scaling. The last requirement (28) is the crucial ingredient of the EUP modification which simply aims to build up the fluctuations whenever $\kappa > 1$. As an easy example, if $w(z) = z^{1/s}$, with $s > 1$, the modified scaling law would give $\Delta N_l \mapsto \kappa^s \Delta N_l$. The nested structure (w^{-1} on top) is necessary since for $\kappa = 1$ the transformation shall be the identity.

We start the discussion of the modified EUP (25) with an observation, that the functions w different from identity break the spherical symmetry of $(\Delta N)^2$. Note however, that if we simply redefine the fluctuation components as $\Delta N_l^{(w)} = w(\Delta N_l)$, we recover the whole structure of the original EUP. The quantity $\Delta N^{(w)}$, defined via the formula (16) with ΔN_l replaced by $\Delta N_l^{(w)}$, not only is inversely correlated with δx , but also enjoys the desired spherical symmetry. The derivation of the formula (20) presented in [16] can thus immediately be repeated for $(\Delta N^{(w)})^2$. Moreover, the function w^{-1} preserves the argumentation based on the independent subsystems, so that $\Delta N_l^{(w)} = \sqrt{CF_l[P]}$. The last observation is valid since in order to make ΔN_l independent (for separable case), also $\Delta N_l^{(w)}$ need (this is also sufficient) to be independent.

As a counterpart of Eq. 21 we obtain

$$\delta x_l \Delta N_l^{(w)} = \frac{\hbar}{2} = \delta x_l w(\Delta N_l). \quad (29)$$

Since the function w is increasing, the relation $\Delta p_l \geq \Delta N_l$ implies $w(\Delta p_l) \geq w(\Delta N_l)$. Directly from (29) we obtain the following, modified uncertainty relation

$$\Delta x_l w(\Delta p_l) \geq \frac{\hbar}{2}, \quad (30)$$

valid, for any l , instead of the traditional HUR. Note that due to (28), this new form of UR is sharper than the HUR.

For instance, if we want (30) to be equal to the GUP, we need to set

$$w(z) = \frac{z}{1 + \beta z^2}. \quad (31)$$

The above function is increasing as long as $z \leq \beta^{-1/2}$, so that the range $0 \leq \Delta p \leq \beta^{-1/2}$ needs to be taken as the domain of validity for our approach in the GUP case. Assuming $\beta_0 = 1$, this

gives a restriction of order $\Delta p \leq \hbar/l_p$, so the upper bound for Δp approximately equal to $10^{23}m_e c$, with m_e being a mass of an electron. Actually, the GUP contribution to HUR can be treated as a correction only under this limitation (otherwise it becomes a dominant term and higher-order corrections need to be taken into account). Only if $\Delta p \leq \beta^{-1/2}$, the superlinear scaling of the momentum fluctuations encompasses the $\beta(\Delta p)^2$ contribution in (1).

The new form of the fluctuation functional, $\Delta N_l = w^{-1}(\sqrt{CF_l[P]})$, after being inserted into the action \mathcal{A}_Q , modifies the quantum Hamilton-Jacobi equation (22) to the form

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + \frac{\hbar^2}{8m} \sum_{l=1}^n (1 + W_l[P]) \left[\frac{1}{P^2} \left(\frac{\partial P}{\partial x_l} \right)^2 - \frac{2}{P} \frac{\partial^2 P}{\partial x_l^2} \right] + V = 0, \quad (32)$$

with

$$W_l[P] = W(CF_l[P]), \quad (33)$$

and the function $W(\cdot)$ equal to

$$W(z) = \frac{d}{dz} [w^{-1}(\sqrt{z})]^2 - 1. \quad (34)$$

The continuity equation (8) once more stays untouched. In our model example of the GUP we easily find that

$$W(z) = \frac{2}{1 + \sqrt{1 - 4\beta z} - 2\beta z(2 + \sqrt{1 - 4\beta z})} - 1 = 4\beta z + \mathcal{O}(\beta^2). \quad (35)$$

The terms proportional to W are naturally responsible for the nonlinear corrections to the Schrödinger equation. The substitution (23), $P = \rho$, and the identity

$$\frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial x_l} \right)^2 - \frac{2}{\rho} \frac{\partial^2 \rho}{\partial x_l^2} = -\frac{4}{|\psi|} \frac{\partial^2 |\psi|}{\partial x_l^2}, \quad (36)$$

bring Eq. 32 to the final form (2).

4. Harmonic oscillator example

We would now briefly like to study eventual manifestations of the modified quantum stationary dynamics. As mentioned in the Introduction the free-particle case, $V = 0$, is trivial as for the plane wave solutions $|\psi|$ is a constant. This fact implicitly implies that the plane waves remain the quantum states with $\Delta p = 0$, even though the whole EUP formalism does not look for the operator definition of momentum.

Let us thus work out the one-dimensional, stationary case of the harmonic potential $V = \frac{1}{2}\zeta x^2$:

$$E\psi = \left(-\frac{\hbar^2(1+\nu)}{2m} \frac{d^2}{dx^2} + \frac{1}{2}\zeta x^2 \right) \psi, \quad (37)$$

with $\nu = W(CF[\rho])$. Since in the textbook solutions for the harmonic oscillator, all the energy eigenstates possess real wave functions (this fact is already included in the above equation), there is no obstacle in performing the complete analysis of this problem. For simplicity, we restrict here however only to the (normalized) ground state given by:

$$\psi_0(x; \nu) = \left(\frac{1}{\pi\sigma^2} \right)^{1/4} e^{-x^2/2\sigma^2}, \quad \sigma^2 = \sigma_0^2 \sqrt{1+\nu}, \quad \sigma_0^2 = \sqrt{\frac{\hbar^2}{\zeta m}}. \quad (38)$$

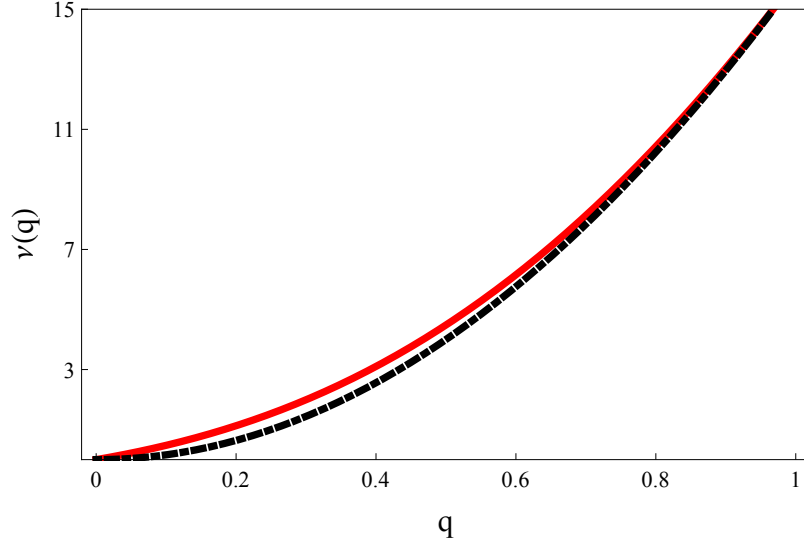


Figure 1. The function $\nu(q)$ (red, solid) together with its asymptotic form $16q^2$ (black, dotted).

The Fisher information of this state is

$$F[\rho_0] = \frac{2}{\sigma^2} \equiv \frac{2}{\sigma_0^2 \sqrt{1 + \nu}}. \quad (39)$$

While focusing on the GUP example, we need to use the function (35), to determine the value of ν . We find:

$$\nu(q) = \frac{q}{1 + q^2} \left(4\sqrt{1 + q^2} + q \left[7 + 8q \left(q + \sqrt{1 + q^2} \right) \right] \right), \quad (40)$$

where $q = \hbar^2 \beta / (2\sigma_0^2)$. This potentially nontrivial function actually very quickly (as can be seen on Fig. 1) becomes indistinguishable from $16q^2$, which represents the asymptotic behavior of $\nu(q)$ for large q . We shall use this fact to anticipate that $\lim_{q \rightarrow \infty} q^{-1} \sqrt{1 + \nu(q)} = 4$.

Obviously for larger values of the basic variance σ_0^2 , the effect of the above modification is negligible. The theory, however nicely predicts the minimal position uncertainty which for the ground state of the harmonic oscillator is given by:

$$\min(\Delta x)^2 = \frac{1}{2} \min_{\sigma_0 \geq 0} \sigma^2 = \frac{1}{2} \min_{\sigma_0 \geq 0} \sigma_0^2 \sqrt{1 + \nu(\hbar^2 \beta / (2\sigma_0^2))}. \quad (41)$$

If we change the variable to q

$$\min(\Delta x)^2 = \frac{\hbar^2 \beta}{4} \min_{q \geq 0} \frac{1}{q} \sqrt{1 + \nu(q)}, \quad (42)$$

and calculate the derivative

$$\frac{d}{dq} \left(\frac{1}{q} \sqrt{1 + \nu(q)} \right) = - \frac{\sqrt{1 + q^2} + 2q \left[1 + q \left(q + \sqrt{1 + q^2} \right) \right]}{q^2 (1 + q^2)^2 \sqrt{8q^2 (1 + q^2) + 1 + 4q \sqrt{1 + q^2} (1 + 2q^2)}} \quad (43)$$

which happens to be always negative (so that the function to be minimized is decreasing), we can conclude that

$$\min (\Delta x)^2 = \frac{\hbar^2 \beta}{4} \lim_{q \rightarrow \infty} q^{-1} \sqrt{1 + \nu(q)} = \hbar^2 \beta \equiv \beta_0 l_p^2. \quad (44)$$

As already mentioned, obtaining the limit is straightforward since for large q one finds $\nu(q) \sim 16q^2$ as only the very last terms from (40) contribute. In the next section we explain why the above observation is universally valid for any solution of the discussed equation.

Exactly the same result, could be obtained by optimizing the GUP with respect to Δp . This known fact follows from

$$\min \Delta x = \hbar \min_{\Delta p \geq 0} \left[\frac{1 + \beta (\Delta p)^2}{2\Delta p} \right] = \hbar \sqrt{\beta}, \quad (45)$$

which is a basic optimization problem.

5. Conclusions

The main achievement of this contribution is the new proposal for the nonlinear Schrödinger equation which enjoys a number of desired properties, namely, separability for non-interacting subsystems, homogeneity and Galilean invariance. It can thus serve as a support for future understanding of the quantum evolution origins and the validity of its linearity. Moreover, this equation has been derived from the generalized form of the exact uncertainty principle, which serves as a prerequisite for the generalized versions of the Heisenberg uncertainty relation, such as the GUP (1).

Using the GUP as a toy model introducing additional gravitational effects to the standard description of quantum uncertainty, we shall point out two important observations. First of all, while thinking about the relevance of the presented nonlinearities (in particular, in the GUP context) one could (and actually should) question the marriage of classical mechanics and extreme quantum regimes necessarily related to the nonlinearities. Even though, on a first impression this match seems to be conceptually problematic, it happens to be able to capture fundamental features of quantum systems under discussion, such as the minimal uncertainty of the position variable. In Sec. 4 this property is shown for the ground state of the harmonic oscillator. It is however of general validity since the function $W(z)$ in (35) is singular for $z = 0$ and $z = 1/(4\beta)$, while it becomes complex whenever $z > 1/(4\beta)$. The first case is not relevant because $z = CF[\rho]$ is equal to 0 only for plane waves, for which the derivatives of $|\psi|$ do vanish anyway. Due to the two remaining issues we necessarily have $F[\rho] \leq 1/(\hbar^2 \beta)$, and the physical system needs to acquire infinite energy to saturate the inequality. This last result, by virtue of the Cramer-Rao bound is equivalent to existence of the minimal observable distance equal to $\hbar \sqrt{\beta}$.

The second observation is the main physical insight of the approach — the fact that plane waves do not feel the nonlinear interaction. Since it is possible to extend the basic EUP formalism to the case of the Klein Gordon equation [31] there is no obstacle to directly apply the results of this paper in the relativistic case. Without going into the details (an extended analysis could be performed in the future) I conclude that gravitational effects added to the standard relativistic dynamics do not need to affect plane-wave solutions. It does not need to be true (even though it is commonly believed [32, 33]) that dispersion relations of the plane waves are affected on the Planck scale thus carrying a signature of quantum gravitational effects. Moreover, the issues with Lorentz invariance could potentially be solved in the same way (relativistic dynamics together with

generalized EUP) if one starts from the relativistic counterpart of the GUP. The last requirement is conceptually a bit difficult. The proposal related to the maximal proper acceleration approach [27] seems to be promising in that context.

Staying with the GUP model I would like to emphasize another particularly valuable content of the formalism presented. In [23], the Schrödinger equation in the GUP case has been derived in the momentum representation. Issues related to the description of the position-momentum duality made it impossible to develop the equation in the position domain. Moreover, the momentum representation equation, while linear, contained momentum derivatives of every order. On the contrary, Eq. 2 stays on the ground of the position representation, and due to its classical roots actually knows nothing about the operator nature of both position and momentum variables. This new evolution equation, while quasi-nonlinear, involves only second order spatial derivatives. It can be thus treated as the opposite side of the mirror, in comparison with the GUP Schrödinger equation from [23].

Last, but not least, there is a single contribution [34] which utilizes the EUP approach to discuss nonlinearities of the Schrödinger dynamics. The resulting equation differs substantially from the one derived here as it again contains derivatives of any order. It could be interesting to have a closer look at interrelations between the two approaches, also in the context of plane waves.

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